

# COMPLEX INTERPOLATION OF COMPACT OPERATORS MAPPING INTO LATTICE COUPLES.

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ABSTRACT. Suppose that  $(A_0, A_1)$  and  $(B_0, B_1)$  are Banach couples, and that  $T$  is a linear operator which maps  $A_0$  compactly into  $B_0$  and  $A_1$  boundedly (or even compactly) into  $B_1$ .

Does this imply that  $T : [A_0, A_1]_\theta \rightarrow [B_0, B_1]_\theta$  compactly?

(Here, as usual,  $[A_0, A_1]_\theta$  denotes the complex interpolation space of Alberto Calderón.)

This question has been open for 44 years. Affirmative answers are known for it in many special cases.

We answer it affirmatively in the case where  $(A_0, A_1)$  is arbitrary and  $(B_0, B_1)$  is a couple of Banach lattices having absolutely continuous norms or the Fatou property.

## 1. INTRODUCTION.

All Banach spaces in this paper will be over the complex field. The closed unit ball of a Banach space  $A$  will be denoted by  $\mathcal{B}_A$ . For any two Banach spaces  $A$  and  $B$ , the notation  $T : A \xrightarrow{b} B$  will mean, just like the usual notation  $T : A \rightarrow B$ , that  $T$  is a linear operator  $T$  defined on  $A$  (and also possibly defined on a larger space) and it maps  $A$  into  $B$  boundedly. The notation  $T : A \xrightarrow{c} B$  will mean that  $T : A \xrightarrow{b} B$  with the additional condition that  $T$  maps  $A$  into  $B$  compactly.

We will write  $A \overset{1}{\subset} B$  when  $A$  is continuously embedded with norm 1 into  $B$ , and  $A \overset{1}{=} B$  when  $A$  and  $B$  coincide with equality of norms.

For each **Banach couple** (or **interpolation pair**)  $\vec{A} = (A_0, A_1)$  and each  $\theta \in [0, 1]$ , we will let  $[A_0, A_1]_\theta$  denote the complex interpolation space of Alberto Calderón [3]. We also let  $A_j^\circ$  denote the closure of  $A_0 \cap A_1$  in  $A_j$  for  $j = 0, 1$ . The couple  $(A_0, A_1)$  is called **regular** if  $A_j^\circ = A_j$  for  $j = 0, 1$ . The spaces  $A_0 \cap A_1$  and  $A_0 + A_1$  are Banach spaces when they are equipped with their usual norms (as e.g., on p. 114 of [3]).

For any two fixed Banach couples  $\vec{A} = (A_0, A_1)$  and  $\vec{B} = (B_0, B_1)$ , the notation  $T : \vec{A} \xrightarrow{c,b} \vec{B}$  will mean that the linear operator  $T : A_0 + A_1 \rightarrow B_0 + B_1$  satisfies  $T : A_0 \xrightarrow{c} B_0$  and  $T : A_1 \xrightarrow{b} B_1$ . The notation  $\vec{A} \blacktriangleright \vec{B}$  will mean that every linear operator  $T : A_0 + A_1 \rightarrow B_0 + B_1$  which satisfies  $T : \vec{A} \xrightarrow{c,b} \vec{B}$  also satisfies  $T : [A_0, A_1]_\theta \xrightarrow{c} [B_0, B_1]_\theta$  for every  $\theta \in (0, 1)$ . The notation  $(*.*) \blacktriangleright \vec{B}$  for some fixed

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Banach couple  $\vec{B}$  will mean that  $\vec{A} \blacktriangleright \vec{B}$  for every Banach couple  $\vec{A}$ . Analogously, the notation  $\vec{A} \blacktriangleright (*.*)$  for some fixed Banach couple  $\vec{A}$  will mean that  $\vec{A} \blacktriangleright \vec{B}$  for every Banach couple  $\vec{B}$ .

Some forty-four years ago, Calderón proved [3] that  $(*,*) \blacktriangleright \vec{B}$  for all Banach couples  $\vec{B}$  which satisfy a certain approximation condition. Since then it has been established that  $\vec{A} \blacktriangleright \vec{B}$  for a large variety of other different choices of  $\vec{A}$  and  $\vec{B}$ . (See, e.g., the twelve papers and website referred to on p. 72 of [7], and [7] itself.) However we still do not know whether  $\vec{A} \blacktriangleright \vec{B}$  holds for *all* choices of  $\vec{A}$  and  $\vec{B}$ . Let us slightly extend the notation just introduced here and rephrase this by saying that we are still trying to discover whether or not  $(*,*) \blacktriangleright (*.*)$ .

In this paper we shall add to the library of known examples of couples  $\vec{A}$  and  $\vec{B}$  satisfying  $\vec{A} \blacktriangleright \vec{B}$  in the context of spaces of measurable functions. We shall use the terminology ***lattice couple*** to mean a Banach couple  $\vec{A} = (A_0, A_1)$  where both  $A_0$  and  $A_1$  are complexified Banach lattices of measurable functions defined on the same  $\sigma$ -finite measure space.

Cobos, Kühn and Schonbek ([4] Theorem 3.2 p. 289) proved that  $\vec{A} \blacktriangleright \vec{B}$  whenever *both*  $\vec{A}$  and  $\vec{B}$  are lattice couples, provided that  $B_0$  and  $B_1$  both have the Fatou property, or that at least one of  $B_0$  and  $B_1$  has absolutely continuous norm. Subsequently, Cwikel and Kalton ([8] Corollary 7 part (c) on p. 270) generalized this result by showing that  $\vec{A} \blacktriangleright (*.*)$  for any lattice couple  $\vec{A}$ .

In this paper we shall obtain a different generalization of the above mentioned result of [4], namely we will show that  $(*,*) \blacktriangleright \vec{B}$  for every lattice couple  $\vec{B} = (B_0, B_1)$  satisfying one or the other of the same conditions imposed by Cobos, Kühn and Schonbek. In fact some other weaker conditions on  $\vec{B}$  are also sufficient. Roughly speaking, as indeed the reader might naturally guess, our approach is to take the “adjoint” of the above mentioned result  $\vec{A} \blacktriangleright (*.*)$  of [8], using arguments in the style of Schauder’s classical theorem about adjoints of operators. But this is apparently not quite as simple to do as one might at first expect. In fact it will be convenient to use a somewhat more general “abstract” version of Schauder’s theorem.

As we will show in a forthcoming paper, our main result here is one of the components needed to show that  $(*,*) \blacktriangleright \vec{B}$  also for certain other couples  $\vec{B}$ , *non* lattice couples which have not been considered previously in this context, and which are in some sense quite close to the couple  $(\ell^\infty(FL^\infty), \ell^\infty(FL_1^\infty))$ . This latter couple is very important and interesting because (cf. Proposition 3 on p. 356 of [9], or [5] pp. 339–340) determining whether or not  $(*,*) \blacktriangleright (\ell^\infty(FL^\infty), \ell^\infty(FL_1^\infty))$  is equivalent to determining whether or not  $(*,*) \blacktriangleright (*.*)$ .

Recently Evgeniy Pustylnik [16] has obtained a very general compactness theorem which in some ways is similar and in some ways more general than ours. In fact his result applies to a wide range of interpolation methods, not just Calderón’s, and the spaces  $B_0$  and  $B_1$  in his “range” couple can be rather more general than Banach lattices. On the other hand they have to satisfy certain conditions which we do not require for our range spaces.

## 2. A RATHER GENERAL ARZELÀ-ASCOLI-SCHAUDER THEOREM.

In this section we describe the result which will play the rôle of Schauder's theorem for the proof of our main result.

Let us recall that a **semimetric space**  $(X, d)$ , also often referred to as a **pseudometric space**, is defined exactly like a metric space, except that the condition  $d(x, y) = 0$  for a pair of points  $x, y \in X$  does not imply that  $x = y$ . (However  $d(x, x) = 0$  for all  $x \in X$ .) Each semimetric space  $(X, d)$  gives rise to a metric space  $(\tilde{X}, \tilde{d})$  in an obvious way, where  $\tilde{X}$  is the set of equivalence classes of  $X$  defined by the relation  $x \sim y \iff d(x, y) = 0$ .

Here are three definitions and three propositions concerning an arbitrary semimetric space  $(X, d)$ . The definitions are exactly analogous to standard definitions for metric spaces, and the propositions are proved exactly analogously to the proofs of the corresponding standard propositions in the case of metric spaces, or by invoking those standard propositions for the particular metric space  $(\tilde{X}, \tilde{d})$ .

**Definition 2.1.** Let  $B(x, r)$  denote the **ball of radius  $r$  centred at  $x$** , i.e., for each  $x \in X$  and  $r > 0$ , we set  $B(x, r) = \{y \in X : d(x, y) \leq r\}$ .

**Definition 2.2.** The semimetric space  $(X, d)$  is said to be **totally bounded** if, for each  $r > 0$ , there exists a finite set  $F_r \subset X$  such that  $X = \bigcup_{x \in F_r} B(x, r)$ .

**Definition 2.3.** The semimetric space  $(X, d)$  is said to be **separable** if there exists a countable set  $Y \subset X$  such that  $\inf_{y \in Y} d(x, y) = 0$  for each  $x \in X$ .

**Proposition 2.4.** If  $(X, d)$  is totally bounded, then it is separable.

**Proposition 2.5.**  $(X, d)$  is not totally bounded if and only if for some  $r > 0$  there exists an infinite set  $E \subset X$  such that  $d(x, y) > r$  for all  $x, y \in E$  with  $x \neq y$ .

**Proposition 2.6.**  $(X, d)$  is totally bounded if and only if every sequence  $\{x_n\}_{n \in \mathbb{N}}$  in  $X$  has a **Cauchy subsequence**, i.e., a subsequence  $\{x_{n_k}\}_{k \in \mathbb{N}}$  which satisfies  $\lim_{N \rightarrow \infty} \sup \{d(x_{n_p}, x_{n_q}) : p, q > N\} = 0$ .

For the benefit of any reader who may happen to find them helpful, we have allowed ourselves the luxury of including the (very standard) proofs of Propositions 2.4, 2.5 and 2.6 in an appendix (Section 6). Since the writer and the reader(s) of this paper can choose to keep it only electronic, we do not have to worry too much about wasting paper.

The following theorem contains the classical theorems of Arzelà-Ascoli and of Schauder. It can be considered as a special case, a “lite” version, of considerably more abstract results presented by Robert G. Bartle in [1] (cf. also e.g., [15]) and which have their roots in earlier work of R. S. Phillips, Šmulian and Kakutani. However it seems easier to give a direct proof of this theorem than to deduce it from [1].

**Theorem 2.7.** Let  $A$  and  $B$  be two sets and let  $h : A \times B \rightarrow \mathbb{C}$  be a function with the properties that

$$\sup_{a \in A} |h(a, b)| < \infty \text{ for each fixed } b \in B, \text{ and} \quad (1)$$

$$\sup_{b \in B} |h(a, b)| < \infty \text{ for each fixed } a \in A. \quad (2)$$

Define  $d_A(a_1, a_2) := \sup_{b \in B} |h(a_1, b) - h(a_2, b)|$  for each pair of elements  $a_1$  and  $a_2$  in  $A$ .

Define  $d_B(b_1, b_2) = \sup_{a \in A} |h(a, b_1) - h(a, b_2)|$  for each pair of elements  $b_1$  and  $b_2$  in  $B$ .

Then

$$(A, d_A) \text{ and } (B, d_B) \text{ are semimetric spaces} \quad (3)$$

and

$$(A, d_A) \text{ is totally bounded if and only if } (B, d_B) \text{ is totally bounded.} \quad (4)$$

*Proof.* It is obvious that (3) holds. For the proof of (4), because of the symmetrical roles of  $A$  and  $B$ , we only have to prove one of the two implications.

Suppose then that  $(A, d_A)$  is totally bounded. By Proposition 2.4, there exists a countable subset  $Y$  of  $A$  which is dense in  $A$ . Let us show that

$$d_B(b_1, b_2) = \sup_{y \in Y} |h(y, b_1) - h(y, b_2)| \text{ for all } b_1, b_2 \in B. \quad (5)$$

The inequality “ $\geq$ ” in (5) is obvious. For the reverse inequality, given any  $b_1$  and  $b_2$  in  $B$  and any arbitrarily small positive  $\epsilon$ , we choose  $a \in A$  such that

$$d_B(b_1, b_2) \leq |h(a, b_1) - h(a, b_2)| + \epsilon/3. \quad (6)$$

Then we choose  $z \in Y$  such that

$$d_A(z, a) < \epsilon/3. \quad (7)$$

$$\begin{aligned} \text{It follows that } & |h(a, b_1) - h(a, b_2)| \\ & \leq |h(a, b_1) - h(z, b_1)| + |h(z, b_1) - h(z, b_2)| + |h(z, b_2) - h(a, b_2)| \\ & \leq 2d_A(a, z) + \sup_{y \in Y} |h(y, b_1) - h(y, b_2)|. \end{aligned}$$

Combining this with (6) and (7), we can immediately complete the proof of (5).

We shall now assume that  $(B, d_B)$  is not totally bounded and show that this leads to a contradiction. By this assumption and by Proposition 2.5, there exists some positive number  $r$  and some infinite sequence  $\{b_n\}_{n \in \mathbb{N}}$  of elements of  $B$  such that

$$d_B(b_m, b_n) > r \text{ for each } m, n \in \mathbb{N} \text{ with } m \neq n. \quad (8)$$

For each fixed  $y \in Y$  it follows from (2) that the numerical sequence  $\{h(y, b_n)\}_{n \in \mathbb{N}}$  is bounded and thus has a convergent subsequence. Since  $Y$  is countable we can apply a standard Cantor “diagonalization” argument to obtain a subsequence  $\{b_{\gamma_n}\}_{n \in \mathbb{N}}$  of  $\{b_n\}_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow \infty} h(y, b_{\gamma_n})$  exists for each  $y \in Y$ . Therefore, after simply changing our notation, we can assume the existence of an infinite sequence  $\{b_n\}_{n \in \mathbb{N}}$  in  $B$  which satisfies (8) and also

$$\lim_{n \rightarrow \infty} h(y, b_n) \text{ exists and is finite for each } y \in Y. \quad (9)$$

In view of (5) and (8), for each pair of integers  $m$  and  $n$  with  $0 < m < n$  there exists an element  $y_{m,n} \in Y$  such that  $|h(y_{m,n}, b_m) - h(y_{m,n}, b_n)| > r$ , and so, in particular,

$$|h(y_{m,m+1}, b_m) - h(y_{m,m+1}, b_{m+1})| > r \text{ for all } m \in \mathbb{N}. \quad (10)$$

Our assumption that  $(A, d_A)$  is totally bounded ensures, by Proposition 2.6, that there exists a strictly increasing sequence of positive integers  $\{m_k\}_{k \in \mathbb{N}}$  such that

$\{y_{m_k, m_k+1}\}_{k \in \mathbb{N}}$  is a Cauchy sequence in  $(A, d_A)$ . Now we set  $z_k = y_{m_k, m_k+1}$  for each  $k$ . We choose some sufficiently large integer  $N$  for which

$$d_A(z_N, z_k) < r/4 \text{ for all } k \geq N. \quad (11)$$

Now we combine (10) and (11) to obtain that, for each  $k \geq N$ ,

$$\begin{aligned} r &< |h(z_k, b_{m_k}) - h(z_k, b_{m_k+1})| \\ &\leq |h(z_k, b_{m_k}) - h(z_N, b_{m_k})| + |h(z_N, b_{m_k}) - h(z_N, b_{m_k+1})| \\ &\quad + |h(z_N, b_{m_k+1}) - h(z_k, b_{m_k+1})| \\ &< \frac{r}{4} + |h(z_N, b_{m_k}) - h(z_N, b_{m_k+1})| + \frac{r}{4}. \end{aligned}$$

In view of (9), we obtain that  $\lim_{k \rightarrow \infty} |h(z_N, b_{m_k}) - h(z_N, b_{m_k+1})| = 0$ . So the inequalities on the preceding lines would imply that  $r \leq r/2$ . This contradiction shows that  $(B, d_B)$  must be totally bounded, and so completes the proof of the theorem.  $\square$

### 3. PRELIMINARIES ABOUT LATTICES AND LATTICE COUPLES.

Let  $(\Omega, \Sigma, \mu)$  be a  $\sigma$ -finite measure space here and in the sequel. (Some of the assertions which we will be making here are simply false if  $(\Omega, \Sigma, \mu)$  is not  $\sigma$ -finite.)

**Definition 3.1.** *We say that a Banach space  $X$  is a **CBL**, or a **complexified Banach lattice of measurable functions on  $\Omega$**  if*

- (i) *all the elements of  $X$  are (equivalence classes of a.e. equal) measurable functions  $f : \Omega \rightarrow \mathbb{C}$  and*
- (ii) *for any measurable functions  $f : \Omega \rightarrow \mathbb{C}$  and  $g : \Omega \rightarrow \mathbb{C}$ , if  $f \in X$  and  $|g| \leq |f|$  a.e., then  $g \in X$  and  $\|g\|_X \leq \|f\|_X$ .*

We will now recall a number of definitions and basic facts about CBLs. In several cases the relevant proofs of these facts in the literature to which we refer are given for Banach lattices of *real* valued functions. But in all those cases it is an obvious and easy exercise to adapt those proofs to our case here.

Any two CBLs  $X_0$  and  $X_1$  on the same underlying measure space always form a Banach couple. See e.g., [3] p. 122 and p. 161 or [13] Corollary 1, p. 42, or [11] Remark 1.41, pp. 34–35. (As explicitly stated and shown in [11], this is also true for non  $\sigma$ -finite measure spaces.)

For each CBL  $X$  on  $(\Omega, \Sigma, \mu)$ , there exists a measurable subset  $\Omega_X$  of  $\Omega$ , which may be called the **support** of  $X$ , such that, for every function  $g \in X$ , we have  $g(\omega) = 0$  for a.e.  $\omega \in \Omega \setminus \Omega_X$ . Furthermore, there exists a function  $f_X \in X$  such that  $f_X(\omega) > 0$  for a.e.  $\omega \in \Omega_X$ . (Cf. e.g., Remarks 1.3 and 1.4 on p. 14 of [11].) Obviously the set  $\Omega_X$  is unique to within a set of measure zero. (Of course, on the other hand, the function  $f_X$  certainly is *not* unique.) If  $\Omega_X = \Omega$  (at least to within a set of measure zero) then we say that  $X$  is **saturated**.

The set  $\Omega_X$  has an additional useful property: There exists a sequence of sets  $\{E_n\}_{n \in \mathbb{N}}$  in  $\Sigma$  such that

$$\Omega_X = \bigcup_{n \in \mathbb{N}} E_n \text{ with } E_n \subset E_{n+1}, \mu(E_n) < \infty, \text{ and } \chi_{E_n} \in X \text{ for each } n \in \mathbb{N}. \quad (12)$$

The actual construction of  $\Omega_X$  and of the sequence  $\{E_n\}_{n \in \mathbb{N}}$  can be performed by an “exhaustion” process described in the proof of Theorem 3 on pp. 455–456

of [18] and also described (perhaps slightly more explicitly for our purposes here) in the first part of the proof of Proposition 4.1 on p. 58 of [10]. (Note however that there is a small misprint in [10], the omission of “ $\mu(E)$ ”, in the third line of this latter proof. I.e., the numbers  $\alpha_k$  must of course be defined by  $\alpha_k = \sup \{\mu(E) : E \in \Sigma, E \subset F_k, \chi_E \in X\}$ .) For one possible (very easy and of course not unique) way to construct a function  $f_X \in X$  with the above mentioned property see, e.g., [11] p. 14 Remark 1.4.

We will say that  $(X_0, X_1)$  is a **saturated lattice couple**, if it is a lattice couple and both  $X_0$  and  $X_1$  are saturated.

**Lemma 3.2.** *If  $(X_0, X_1)$  is a saturated lattice couple then  $X_0 \cap X_1$  is saturated, and  $[X_0, X_1]_\theta$  is saturated for each  $\theta \in (0, 1)$ .*

*Proof.* Let  $(\Omega, \Sigma, \mu)$  be the underlying measure space for the couple. The function  $\min\{f_{X_0}, f_{X_1}\}$  is in  $X_0 \cap X_1$  and therefore it is also in  $[X_0, X_1]_\theta$ . It is strictly positive a.e. on  $\Omega$ . So neither of the sets  $\Omega \setminus \Omega_{X_0 \cap X_1}$  and  $\Omega \setminus \Omega_{[X_0, X_1]_\theta}$  can have positive measure.  $\square$

Given an arbitrary CBL  $X$  on  $(\Omega, \Sigma, \mu)$  we define the functional  $\|\cdot\|_{X'}$  by

$$\|f\|_{X'} := \sup \left\{ \left| \int_{\Omega} f g d\mu \right| : g \in X, \|g\|_X \leq 1 \right\} \quad (13)$$

for each measurable function  $f : \Omega \rightarrow \mathbb{C}$ .

**Remark 3.3.** *Obviously we can replace  $\left| \int_{\Omega} f g d\mu \right|$  by  $\int_{\Omega} |f g| d\mu$  in the formula (13).*

Let  $X'$  be the set of all measurable functions  $f : \Omega \rightarrow \mathbb{C}$  for which  $\|f\|_{X'} < \infty$ . Clearly  $X'$  is a linear space and  $\|\cdot\|_{X'}$  is a seminorm on  $X'$  satisfying

$$\left| \int_{\Omega} f g d\mu \right| \leq \|f\|_{X'} \|g\|_X \text{ for all } f \in X' \text{ and all } g \in X. \quad (14)$$

The space  $X'$  is customarily referred to as the **Köthe dual** or the **associate space** of  $X$ .

If  $\mu(\Omega_X) > 0$ , then, via a series of theorems, including one ([18] Theorem 1, p. 470) which uses Hilbert space techniques, it can be shown that  $X'$  is non trivial, i.e., it contains elements which do not vanish a.e. on  $\Omega_X$ . If, furthermore,  $X$  is saturated, then  $\|\cdot\|_{X'}$  is a norm with respect to which  $X'$  is a saturated CBL on  $(\Omega, \Sigma, \mu)$ . (See e.g., [18] p. 472, Theorem 4.)

Of course  $X'$  can be identified with a subspace of  $X^*$ , the dual space of  $X$ , and in some, but not all, cases it is also a **norming subspace** of  $X^*$ , i.e., it satisfies

$$\|g\|_X = \sup \left\{ \left| \int_{\Omega} f g d\mu \right| : f \in X', \|f\|_{X'} \leq 1 \right\} \text{ for each } g \in X. \quad (15)$$

The following result of Lorentz and Luxemburg, which appears as Proposition 1.b.18 on p. 29 of [14], gives necessary and sufficient conditions on  $X$  for (15) to hold. In particular, it implies that the  $\sigma$ -order continuity of  $X$  is a sufficient condition. So is the Fatou property.

**Theorem 3.4.** *Let  $X$  be an arbitrary CBL on a  $\sigma$ -finite measure space  $(\Omega, \Sigma, \mu)$ . The associate space  $X'$  is a norming subspace of  $X^*$  if and only if  $\lim_{n \rightarrow \infty} \|f_n\|_X = \|f\|_X$  for every non negative function  $f \in X$  and every sequence  $\{f_n\}_{n \in \mathbb{N}}$  of measurable functions satisfying  $0 \leq f_n(\omega) \leq f_{n+1}(\omega) \leq f(\omega)$  for a.e.  $\omega$  and each  $n \in \mathbb{N}$  and  $\lim_{n \rightarrow \infty} f_n(\omega) = f(\omega)$  for a.e.  $\omega$ .*

*Proof.* The proof is essentially the same as the proof of Proposition 1.b.18 on p. 29 of [14]. Some small modifications are required in lines 6 and 7 of that proof on p. 30 of [14], because in the statement and proof of that proposition it is assumed that  $X$  is a Köthe function space. Here is the required replacement for those two lines, using the notation of [14]:

Let now  $f \in X$  be an element with  $\|f\| > 1$ . Using Remark 1.3 on p. 14 of [11], we know that there exists an expanding sequence  $\{E_k\}_{k \in \mathbb{N}}$  of measurable subsets of  $\Omega$  such that  $\chi_{E_k} \in X$  and  $\mu(E_k) < \infty$  for each  $k$ , and such that  $f$  vanishes a.e. on the complement of  $\bigcup_{k \in \mathbb{N}} E_k$ . Then  $|f| \chi_{E_k} \uparrow |f|$  a.e. and so  $\|f \chi_\sigma\| > 1$  for some set  $\sigma = E_k$  for some sufficiently large  $k$ . By using the separation theorem for  $Y$ ....

From here the proof can be continued and concluded exactly as in [14].  $\square$

The associate space  $(X')'$  of  $X'$ , i.e. the *second associate* of  $X$  is usually denoted by  $X''$ . Obviously  $X \subset X''$  and  $\|x\|_{X''} \leq \|x\|_X$  for each  $x \in X$ . Obviously  $X''$  is a CBL whenever  $X$  (and therefore also  $X'$ ) is saturated.

As in e.g., [13], we say that the CBL  $X$  has ***absolutely continuous norm*** if  $\lim_{n \rightarrow \infty} \|f \chi_{E_n}\|_X = 0$  for every  $f \in X$  and every sequence  $\{E_n\}_{n \in \mathbb{N}}$  of measurable sets satisfying  $E_{n+1} \subset E_n$  for all  $n$  and  $\bigcap_{n \in \mathbb{N}} E_n = \emptyset$ . As in e.g., [14], we say that the CBL  $X$  is  ***$\sigma$ -order continuous*** if  $\lim_{n \rightarrow \infty} \|f_n\|_X = 0$  for every sequence  $\{f_n\}_{n \in \mathbb{N}}$  of functions in  $X$  satisfying  $0 \leq f_{n+1} \leq f_n$  and  $\lim_{n \rightarrow \infty} f_n = 0$  a.e. It is easy to see that these two properties of  $X$  are in fact equivalent.

A CBL  $X$  is said to have the ***Fatou property*** if whenever  $\{f_n\}_{n \in \mathbb{N}}$  is a norm bounded a.e. monotonically non decreasing sequence of non negative functions in  $X$ , its a.e. pointwise limit  $f$  is also in  $X$  with  $\|f\|_X = \lim_{n \rightarrow \infty} \|f_n\|_X$ . If  $X$  is saturated, then  $X$  has the Fatou property if and only if  $X = X''$  isometrically. (See [18] p. 472. Cf. also [14] p. 30, but recall that there extra hypotheses are imposed.)

We remark that obvious counterexamples (see e.g., [11] Remark 7.3 p. 92) show that the above claims about  $X'$  and  $X''$  are false for certain non  $\sigma$ -finite measure spaces.

Given a pair of CBLs  $X_0$  and  $X_1$  on  $(\Omega, \Sigma, \mu)$  and a number  $\theta \in (0, 1)$ , we define the space  $X_0^{1-\theta} X_1^\theta$ , analogously to the definition in [3] Section 13.5 pp. 123, to be the set of all measurable functions  $f : \Omega \rightarrow \mathbb{C}$  of the form

$$f = u f_0^{1-\theta} f_1^\theta \quad (16)$$

where  $u \in L^\infty(\mu)$  and  $f_j$  is a non negative function in  $\mathcal{B}_{X_j}$  for  $j = 0, 1$ . For each  $f \in X_0^{1-\theta} X_1^\theta$  we define  $\|f\|_{X_0^{1-\theta} X_1^\theta} = \inf \|u\|_{L^\infty(\mu)}$ , where the infimum is taken over all representations of  $f$  of the form (16) with the stated properties. It can be shown that this is in fact a norm on  $X_0^{1-\theta} X_1^\theta$ , with respect to which  $X_0^{1-\theta} X_1^\theta$  is a CBL. This is proved in Section 33.5 on pp. 164–165 of [3].

The norm 1 inclusions

$$[X_0, X_1]_\theta \stackrel{1}{\subset} X_0^{1-\theta} X_1^\theta \stackrel{1}{\subset} [X_0, X_1]^\theta \quad (17)$$

are special cases (set  $B_0 = B_1 = \mathbb{C}$ ) of the results (i) and (ii) of Section 13.6 on p. 125 of [3] (proved in [3] Section 33.6 on pp. 171–180). Furthermore, with the help of Bergh's theorem [2], (17) can be strengthened to tell us that

$$\|x\|_{[X_0, X_1]_\theta} = \|x\|_{X_0^{1-\theta} X_1^\theta} = \|x\|_{[X_0, X_1]^\theta} \quad \text{for all } x \in [X_0, X_1]_\theta \quad (18)$$

We will need to use the formula

$$(X_0^{1-\theta} X_1^\theta)' = (X_0')^{1-\theta} (X_1')^\theta \quad (19)$$

which holds with equality of norms (or seminorms when  $\Omega_{X_0}$  or  $\Omega_{X_1}$  is strictly smaller than  $\Omega$ ) for all pairs of CBLs  $X_0$  and  $X_1$  on  $(\Omega, \Sigma, \mu)$ . This formula was originally stated and proved by Lozanovskii under certain hypotheses, then by Reisner [17] under other hypotheses. The general version stated here is proved in [11] Section 7, pp. 91–97 using Reisner’s proof and a remark of Kalton.

We will also need this lemma:

**Lemma 3.5.** *Let  $(X_0, X_1)$  be an arbitrary saturated lattice couple. For each  $\theta \in (0, 1)$  we have*

$$\sup_{y \in \mathcal{B}_{[X'_0, X'_1]_\theta}} \left| \int_{\Omega} xy d\mu \right| = \sup_{y \in \mathcal{B}_{(X'_0)^{1-\theta}(X'_1)^\theta}} \left| \int_{\Omega} xy d\mu \right| \text{ for each } x \in X_0 \cap X_1. \quad (20)$$

*Proof.* Applying (18) to the couple  $(X'_0, X'_1)$ , we of course obtain the inequality “ $\leq$ ” in (20). To show the reverse inequality “ $\geq$ ”, we fix some  $x \in X_0 \cap X_1$  and  $y \in \mathcal{B}_{(X'_0)^{1-\theta}(X'_1)^\theta}$  and we shall construct a sequence  $\{y_n\}_{n \in \mathbb{N}}$  in  $\mathcal{B}_{[X'_0, X'_1]_\theta}$  for which

$$\lim_{n \rightarrow \infty} \int_{\Omega} xy_n d\mu = \int_{\Omega} xy d\mu. \quad (21)$$

By Lemma 3.2, since  $X'_0$  and  $X'_1$  are both saturated, so is  $[X'_0, X'_1]_\theta$ . Consequently (cf. (12)) there exists an expanding sequence  $\{E_n\}_{n \in \mathbb{N}}$  of sets in  $\Sigma$  such that  $\Omega = \bigcup_{n \in \mathbb{N}} E_n$  and  $\chi_{E_n} \in [X'_0, X'_1]_\theta$  for each  $n \in \mathbb{N}$ . Let  $y_n = y \chi_{E_n \cap \{\omega \in \Omega : |y(\omega)| \leq n\}}$ . Then  $y_n \in [X'_0, X'_1]_\theta$  and we have  $|xy_n| \leq |xy|$  and  $\lim_{n \rightarrow \infty} x(\omega)y_n(\omega) = x(\omega)y(\omega)$  for all  $\omega \in \Omega$ . The function  $xy$  is integrable, since  $X_0 \cap X_1 \subset [X_0, X_1]_\theta \stackrel{1}{\subset} X_0^{1-\theta} X_1^\theta$  and  $(X'_0)^{1-\theta}(X'_1)^\theta \stackrel{1}{=} (X_0^{1-\theta} X_1^\theta)'$  (cf. (18) and (19) and Remark 3.3). So (21) follows from the Lebesgue dominated convergence theorem. This completes the proof.  $\square$

#### 4. THE MAIN RESULT.

Our main result is a corollary of the following theorem.

**Theorem 4.1.** *Let  $\vec{G} = (G_0, G_1)$  be an arbitrary Banach couple and let  $\vec{X} = (X_0, X_1)$  be a saturated lattice couple. Then every linear operator  $T$  which satisfies  $T : \vec{G} \xrightarrow{c,b} \vec{X}$  has the compactness property*

$$T : [G_0, G_1]_\theta \xrightarrow{c} [X''_0, X''_1]_\theta \quad (22)$$

for each  $\theta \in (0, 1)$ .

**Corollary 4.2.** *Let  $\vec{X} = (X_0, X_1)$  be a saturated lattice couple. Suppose that either*  
*(i)  $X_0$  and  $X_1$  both have the Fatou property, or*  
*(ii) at least one of the spaces  $X_0$  and  $X_1$  is  $\sigma$ -order continuous, or*  
*(iii) there exists at least one value of  $\theta \in (0, 1)$  for which  $X = X_0^{1-\theta} X_1^\theta$  satisfies the condition of Theorem 3.4.*

Then

$$(*, *) \blacktriangleright \vec{X}.$$

**Remark 4.3.** *The requirement that  $(X_0, X_1)$  is saturated is merely a technical convenience which makes the formulation and proof of Theorem 4.1 simpler and shorter. In fact it is entirely unnecessary for Corollary 4.2. The easy and rather obvious argument which extends the proof of Corollary 4.2 to the non saturated*



case uses the easily checked fact that  $\Omega_{X_0 \cap X_1} = \Omega_{[X_0, X_1]_\theta}$  and replaces the spaces  $X_0$  and  $X_1$  in an appropriate way by their “restrictions” to the smaller measure space  $\Omega_{X_0 \cap X_1}$ . The details of that argument are given below in Subsection 5.1.

*Proof of Theorem 4.1.* Since  $[G_0^\circ, G_1^\circ]_\theta = [G_0, G_1]_\theta$  ([3] Sections 9.3 (p. 116) and 29.3 (pp. 113–4)) we can clearly suppose without loss of generality that  $\vec{G}$  is a regular couple. Let  $\langle \cdot, \cdot \rangle$  denote the duality between  $G_0 \cap G_1$  and  $(G_0 \cap G_1)^*$ . Let  $G$  be any one of the spaces  $G_0$ ,  $G_1$  or  $[G_0, G_1]_\theta$  and define  $G^\#$  to be the subspace of elements  $\gamma \in (G_0 \cap G_1)^*$  for which the norm  $\|\gamma\|_{G^\#} := \sup \{ |\langle g, \gamma \rangle| : g \in \mathcal{B}_G \cap G_0 \cap G_1 \}$  is finite. Of course  $G^\#$ , when equipped with this norm, is a Banach space which is continuously embedded in  $(G_0 \cap G_1)^*$ . So  $(G_0^\#, G_1^\#)$  is a Banach couple.

We could of course identify  $G^\#$  with the dual of  $G$ , but it is more convenient to use the above definition. Note also that in fact  $G_0^\# + G_1^\# \stackrel{1}{=} (G_0 \cap G_1)^*$ . Calderón’s remarkable duality theorem ([3] Section 12.1 p. 121 and Section 32.1 pp. 148–156) can be expressed by the formula  $([G_0, G_1]_\theta)^\# \stackrel{1}{=} [G_0^\#, G_1^\#]_\theta$ . For a more detailed discussion of all these issues we refer to [6].

Let  $T$  be an arbitrary linear operator satisfying  $T : \vec{G} \xrightarrow{c,b} \vec{X}$ . We may suppose, without loss of generality, that  $\|T\|_{\vec{G} \rightarrow \vec{X}} := \max_{j=0,1} \|T\|_{G_j \rightarrow X_j} = 1$ . For  $j = 0, 1$ , let  $X'_j$  be the associate space of  $X_j$ . Let  $(\Omega, \Sigma, \mu)$  be the underlying measure space for  $(X_0, X_1)$ . For each  $g \in G_0 \cap G_1$  and each  $z \in X'_0 + X'_1$  define  $h(g, z) = \int_\Omega z T g d\mu$ . Of course (cf. (14)) the function  $h$  satisfies

$$|h(g, z)| \leq \|z\|_{X'_j} \|Tg\|_{X_j} \leq \|z\|_{X'_j} \|g\|_{G_j} \quad (23)$$

for  $j = 0, 1$  and all  $g \in G_0 \cap G_1$  and  $z \in X'_j$ . Therefore  $h$  also satisfies

$$|h(g, z)| \leq \|z\|_{X'_0 + X'_1} \|g\|_{G_0 \cap G_1} \text{ for all } g \in G_0 \cap G_1 \text{ and } z \in X'_0 + X'_1. \quad (24)$$

For each fixed  $z \in X'_0 + X'_1$  we define the linear functional  $Sz$  on  $G_0 \cap G_1$  by  $\langle g, Sz \rangle = h(g, z)$ . Of course  $Sz$  depends linearly on  $z$  and it is clear from (24) that we have thus defined a bounded linear operator  $S : X'_0 + X'_1 \rightarrow (G_0 \cap G_1)^*$ . For  $j = 0, 1$ , in view of (23), we see that, for each  $z \in X'_j$ , we have  $Sz \in G_j^\#$  with  $\|Sz\|_{G_j^\#} \leq \|z\|_{X'_j}$ , i.e.,  $S : X'_j \xrightarrow{b} G_j^\#$ . (Note, cf. [6], that we do not have to consider the extension of  $Sz$  to a space larger than  $G_0 \cap G_1$ .)

We now wish to show that  $S$  satisfies the compactness condition

$$S : X'_0 \xrightarrow{c} G_0^\#. \quad (25)$$

We will do this by applying Theorem 2.7. We consider the restriction of the function  $h(g, y)$  to the set  $A \times B$  where  $A = \mathcal{B}_{G_0} \cap G_1$  and  $B = \mathcal{B}_{X'_0}$ . Given any sequence  $\{g_n\}_{n \in \mathbb{N}}$  in  $A$ , we of course have (cf. (14) and (23)) that  $d_A(g_m, g_n) = \sup \{ |h(g_m, z) - h(g_n, z)| : z \in B \} \leq \|Tg_m - Tg_n\|_{X_0}$ . So the fact that  $T : G_0 \xrightarrow{c} X_0$  implies that  $(A, d_A)$  is totally bounded. (Cf., e.g., Proposition 2.5 or Theorem 15 on p. 22 of [12].) Consequently, in view of Theorem 2.7 and Proposition 2.6, if  $\{z_n\}_{n \in \mathbb{N}}$  is an arbitrary sequence in  $B$ , then it has a subsequence which is Cauchy with respect to the semimetric

$$\begin{aligned} d_B(y, z) &= \sup \{ |h(g, y) - h(g, z)| : g \in A \} \\ &= \sup \{ |\langle g, S(y - z) \rangle| : g \in G_0 \cap G_1, \|g\|_{G_0} \leq 1 \} \\ &= \|S(y - z)\|_{G_0^\#} \end{aligned}$$

This is exactly the condition (25).

Since  $X'_0$  and  $X'_1$  are both CBLs of measurable functions on the measure space  $(\Omega, \Sigma, \mu)$ , we can use (25) and  $S : X'_1 \xrightarrow{b} G_1^\#$  and apply part (c) of Corollary 7 on p. 270 of [8] to deduce that

$$S : [X'_0, X'_1]_\theta \xrightarrow{c} [G_0^\#, G_1^\#]_\theta. \quad (26)$$

We are now ready for a second application of Theorem 2.7. Once more we will use the same function  $h$  defined above and restricted to a set  $A \times B$ , where this time we choose  $A = \mathcal{B}_{[G_0, G_1]_\theta} \cap G_0 \cap G_1$  and  $B = \mathcal{B}_{[X'_0, X'_1]_\theta}$ . This time, for each  $y, z \in B$ , we of course have  $S(y - z) \in [G_0^\#, G_1^\#]_\theta$ . So, using the isometry  $([G_0, G_1]_\theta)^\# \stackrel{1}{=} [G_0^\#, G_1^\#]_\theta$  mentioned above, and then Bergh's theorem [2], we obtain that

$$\begin{aligned} d_B(y, z) &= \sup \{ |\langle g, S(y - z) \rangle| : g \in \mathcal{B}_{[G_0, G_1]_\theta} \cap G_0 \cap G_1 \} = \|S(y - z)\|_{([G_0, G_1]_\theta)^\#} \\ &= \|S(y - z)\|_{[G_0^\#, G_1^\#]_\theta} = \|S(y - z)\|_{[G_0^\#, G_1^\#]_\theta}. \end{aligned}$$

The compactness property (26) of  $S$  implies that  $(B, d_B)$  is totally bounded. Consequently, by Theorem 2.7,  $(A, d_A)$  is also totally bounded. In view of Proposition 2.6 and the fact that  $G_0 \cap G_1$  is dense in  $[G_0, G_1]_\theta$  ([3] Section 9.3 (p. 116) and Section 29.3 (pp. 113–4)), this means that the proof of Theorem 4.1 will be complete once we have shown that

$$d_A(g_1, g_2) = \|Tg_1 - Tg_2\|_{[X''_0, X''_1]_\theta} \text{ for all } g_1, g_2 \in A. \quad (27)$$

By definition, for each  $g_1$  and  $g_2$  in  $A$  we have

$$d_A(g_1, g_2) = \sup_{y \in \mathcal{B}_{[X'_0, X'_1]_\theta}} \left| \int_\Omega y T(g_1 - g_2) d\mu \right|$$

At this stage we do not need to consider the particular form of the element  $Tg_1 - Tg_2$ . We know that it is an element of  $X_0 \cap X_1$ . So, to obtain (27) it suffices to show that

$$\sup_{y \in \mathcal{B}_{[X'_0, X'_1]_\theta}} \left| \int_\Omega xy d\mu \right| = \|x\|_{[X''_0, X''_1]_\theta} \text{ for each } x \in X_0 \cap X_1. \quad (28)$$

Since  $X_0 \cap X_1 \subset X''_0 \cap X''_1 \subset [X''_0, X''_1]_\theta$ , we have, from (18) applied to the couple  $(X''_0, X''_1)$ , that the right side of (28) equals  $\|x\|_{(X''_0)^{1-\theta}(X''_1)^\theta}$  and this in turn, in view of (19) applied to the couple  $(X'_0, X'_1)$ , equals  $\|x\|_{((X'_0)^{1-\theta}(X'_1)^\theta)'} = \sup_{y \in \mathcal{B}_{(X'_0)^{1-\theta}(X'_1)^\theta}} \left| \int_\Omega xy d\mu \right|$ . Thus we can complete the proof by applying Lemma 3.5.  $\square$

*Proof of Corollary 4.2.* As in the preceding proof, we consider an arbitrary regular couple  $(G_0, G_1)$  and an arbitrary operator  $T : \vec{G} \xrightarrow{c,b} \vec{X}$ . We need to show that

$$T : [G_0, G_1]_\theta \xrightarrow{c} [X_0, X_1]_\theta \quad (29)$$

for all  $\theta \in (0, 1)$ . If condition (i) holds then  $X''_0 = X_1$  and  $X''_1 = X_1$  and (22) gives us the required conclusion. Otherwise we simply work through all the same steps as in the preceding proof until we reach (27). Then (29) will follow if, instead of (27), we can establish a variant of (27) or of (28), namely that  $\sup_{y \in \mathcal{B}_{[X'_0, X'_1]_\theta}} \left| \int_\Omega xy d\mu \right| =$

$\|x\|_{[X_0, X_1]_\theta}$  for each  $x \in X_0 \cap X_1$ . Using Lemma 3.5 and then (19) and also (18), we see that the required condition is equivalent to

$$\sup_{y \in \mathcal{B}_{(X_0^{1-\theta} X_1^\theta)'}} \left| \int_{\Omega} xy d\mu \right| = \|x\|_{X_0^{1-\theta} X_1^\theta} \text{ for each } x \in X_0 \cap X_1. \quad (30)$$

In fact condition (ii) implies condition (iii) because, if  $X_0$  or  $X_1$  is  $\sigma$ -order continuous, then  $X_0^{1-\theta} X_1^\theta$  is  $\sigma$ -order continuous for every  $\theta \in (0, 1)$  (cf. Proposition 4 on p. 80 of [17] or Theorem 1.29 on p. 27 of [11]). Condition (iii) ensures that  $(X_0^{1-\theta} X_1^\theta)'$  is a norming subspace of  $(X_0^{1-\theta} X_1^\theta)^*$  for at least one value of  $\theta$ . This implies that (30) holds for that value of  $\theta$ . Consequently, (29) holds for that same value of  $\theta$ . So, by “extrapolation” (see Theorem 2.1, on p. 339 of [5] or Theorem 5.3 on p. 311 of [4]), we obtain (29) for *all*  $\theta \in (0, 1)$ .  $\square$

## 5. FURTHER POSSIBLE GENERALIZATIONS.

**5.1. Extending the result to the case where  $(X_0, X_1)$  is not saturated.** Here we give a more detailed (perhaps too detailed?) explanation of the claim made in Remark 4.3.

**Corollary 5.1.** *The result of Corollary 4.2 also holds if  $X_0$  is not saturated and/or  $X_1$  is not saturated.*

*Proof.* Since  $\Omega_{X_0 \cap X_1} = \{\omega \in \Omega : f_{X_0 \cap X_1}(\omega) > 0\}$  for some non negative function  $f_{X_0 \cap X_1}$  which is in  $X_0 \cap X_1$  and therefore also in  $[X_0, X_1]_\theta$ , we see that

$$\mu(\Omega_{X_0 \cap X_1} \setminus \Omega_{[X_0, X_1]_\theta}) = \mu(\{\omega \in \Omega : f_{X_0 \cap X_1}(\omega) > 0, \omega \notin \Omega_{[X_0, X_1]_\theta}\}) = 0.$$

Next we remark that  $\Omega_{[X_0, X_1]_\theta} = \{\omega \in \Omega : f_{[X_0, X_1]_\theta}(\omega) > 0\}$  for some non negative function  $f_{[X_0, X_1]_\theta}$  which is in  $[X_0, X_1]_\theta$  and therefore in  $X_0^{1-\theta} X_1^\theta$ . We thus have  $f_{[X_0, X_1]_\theta} = f_0^{1-\theta} f_1^\theta$  where  $f_0$  and  $f_1$  are non negative functions in  $X_0$  and  $X_1$  respectively. Clearly

$$v := \min\{f_0, f_1\} \quad (31)$$

is a non negative function in  $X_0 \cap X_1$  and

$$\begin{aligned} \mu(\Omega_{[X_0, X_1]_\theta} \setminus \Omega_{X_0 \cap X_1}) &= \mu(\{\omega \in \Omega : f_{[X_0, X_1]_\theta}(\omega) > 0, \omega \notin \Omega_{X_0 \cap X_1}\}) \\ &= \mu(\{\omega \in \Omega : v(\omega) > 0, \omega \notin \Omega_{X_0 \cap X_1}\}) = 0. \end{aligned}$$

Thus we have shown that

$$\Omega_{X_0 \cap X_1} = \Omega_{[X_0, X_1]_\theta} \text{ a.e.} \quad (32)$$

Now we consider the measure space  $(\Omega_*, \Sigma_*, \mu_*)$  where  $\Omega_* = \Omega_{X_0 \cap X_1}$  and  $\Sigma_*$  is the  $\sigma$ -algebra of all sets in  $\Sigma$  which are contained in  $\Omega_*$ , and  $\mu_*$  is the restriction of  $\mu$  to  $\Sigma_*$ . We shall conveniently “navigate” between spaces of functions on  $\Omega$  and spaces of function on  $\Omega_*$  with the help of two simple and obvious operators  $\mathcal{R}$  and  $\mathcal{E}$  of restriction and extension. For each function  $f : \Omega \rightarrow \mathbb{C}$  let  $\mathcal{R}f$  be the restriction of  $f$  to  $\Omega_*$ . For each function  $g : \Omega_* \rightarrow \mathbb{C}$  let  $\mathcal{E}g$  be the complex valued function on  $\Omega$  which equals 0 on  $\Omega \setminus \Omega_*$  and coincides with  $g$  on  $\Omega_*$ . For  $j = 0, 1$  we let  $Y_j = \mathcal{R}X_j$ . Thus  $Y_j$  is a space of  $\mu_*$  measurable functions  $y : \Omega_* \rightarrow \mathbb{C}$  and we may norm it by setting  $\|y\|_{Y_j} = \|\mathcal{E}y\|_{X_j}$ . It is clear that  $Y_j$  is a CBL. Furthermore it is saturated, because the function  $\mathcal{R}v$  (where  $v$  is the function introduced in (31)) is in  $Y_j$  and is strictly positive a.e. on  $\Omega_*$ . Obviously  $\mathcal{E} : Y_j \xrightarrow{b} X_j$  and  $\mathcal{R} : X_j \xrightarrow{b} Y_j$  for  $j = 0, 1$

with norm 1 in each case. It follows by interpolation that  $\mathcal{E} : [Y_0, Y_1]_\theta \xrightarrow{b} [X_0, X_1]_\theta$  and  $\mathcal{R} : [X_0, X_1]_\theta \xrightarrow{b} [Y_0, Y_1]_\theta$  (also in fact with norm 1).

Suppose now that given couple  $(X_0, X_1)$  satisfies one (or more) of the conditions (i), (ii) and (iii) stated in Corollary 4.2. Then the couple  $(Y_0, Y_1)$  satisfies the same condition: If  $X_j$  has the Fatou property then so, obviously does  $Y_j$ . Also, if  $X_j$  is  $\sigma$ -order continuous, so is  $Y_j$ . To deal with condition (iii) we first remark that it is easy to see that  $\|g\|_{Y_0^{1-\theta}Y_1^\theta} = \|\mathcal{E}g\|_{X_0^{1-\theta}X_1^\theta}$  for each  $g \in Y_0^{1-\theta}Y_1^\theta$ . Consequently, if  $X_0^{1-\theta}X_1^\theta$  satisfies the hypotheses of Theorem 3.4 for some value of  $\theta$ , then so does  $Y_0^{1-\theta}Y_1^\theta$ .

Let  $\vec{G}$  be an arbitrary Banach couple and suppose that  $T : \vec{G} \xrightarrow{c,b} \vec{X}$ . Then the composed operator  $\mathcal{R}T$  satisfies  $\mathcal{R}T : \vec{G} \xrightarrow{c,b} \vec{Y}$ . Thus, by Corollary 4.2, we have  $\mathcal{R}T : [G_0, G_1]_\theta \xrightarrow{c} [Y_0, Y_1]_\theta$ . Consequently

$$\mathcal{E}\mathcal{R}T : [G_0, G_1]_\theta \xrightarrow{c} [X_0, X_1]_\theta. \quad (33)$$

In view of (32), each function  $f \in [X_0, X_1]_\theta$  vanishes a.e. on  $\Omega \setminus \Omega_*$ , and therefore satisfies  $\mathcal{E}\mathcal{R}f = f$ . This means that  $\mathcal{E}\mathcal{R}Tg = Tg$  for each  $g \in [G_0, G_1]_\theta$ . So (33) gives us that  $T : [G_0, G_1]_\theta \xrightarrow{c} [X_0, X_1]_\theta$  and completes the proof of Corollary 5.1.  $\square$

**5.2. Two sided interpolation of compactness.** We may use the notation  $T : \vec{A} \xrightarrow{c,c} \vec{B}$  to mean that the linear operator  $T : A_0 + A_1 \rightarrow B_0 + B_1$  satisfies the “two sided” compactness condition  $T : A_j \xrightarrow{c} B_j$  for *both* values 0 and 1 of  $j$ . Not only do we still not know whether in general  $T : \vec{A} \xrightarrow{c,b} \vec{B}$  implies that  $T : [A_0, A_1]_\theta \xrightarrow{c} [B_0, B_1]_\theta$ , but we cannot even deduce that  $T : [A_0, A_1]_\theta \xrightarrow{c} [B_0, B_1]_\theta$  when  $T$  satisfies the stronger condition  $T : \vec{A} \xrightarrow{c,c} \vec{B}$ . We conjecture however that  $T : \vec{A} \xrightarrow{c,c} \vec{B}$  implies  $T : [A_0, A_1]_\theta \xrightarrow{c} [B_0, B_1]_\theta$  for each  $\theta \in (0, 1)$  for arbitrary couples  $\vec{A}$  whenever  $\vec{B}$  is an arbitrary lattice couple, i.e., with no requirements of Fatou property or  $\sigma$ -order continuity.

**5.3. What if the underlying measure space is not  $\sigma$ -finite?** We have assumed throughout this paper that the underlying measure spaces of our lattice couples are  $\sigma$ -finite. Could there be an exotic counterexample in the realm of non  $\sigma$ -finite measure spaces which would finally settle the question of whether or not  $(*,*) \blacktriangleright (*,*)$ ? This somehow seems unlikely. For a start we can assert that  $\vec{A} \blacktriangleright (*,*)$  also when the underlying measure space of the lattice couple  $\vec{A}$  is not  $\sigma$ -finite. Let us note that the proof in [8] that  $\vec{A} \blacktriangleright (*,*)$  for all lattice couples  $\vec{A}$  (Corollary 7 part (c) on p. 270) assumes, albeit quite implicitly, because of the auxiliary results which it uses, that the underlying measure space is  $\sigma$ -finite. However it is not very difficult to obtain appropriate variants of those auxiliary results for the case of an arbitrary underlying measure space. We plan to provide the details of that in forthcoming paper(s), where we will also present partial results showing that  $(*,*) \blacktriangleright \vec{B}$  for certain lattice couples  $\vec{B}$  which do not satisfy the hypotheses imposed here, including some which are defined on non  $\sigma$ -finite measure spaces.

6. APPENDIX-STANDARD PROOFS OF THE STANDARD PROPOSITIONS 2.4, 2.5  
AND 2.6.

*Proof of Proposition 2.4.* The fact that  $(X, d)$  is totally bounded means that, for each  $k \in \mathbb{N}$ , there exists a finite set of points  $F_k$  such that  $X = \bigcup_{y \in F_k} B(y, 2^{-k})$ . The set  $Y := \bigcup_{k \in \mathbb{N}} F_k$  is of course countable, and, for each  $x \in X$ , we have  $\inf_{y \in Y} d(x, y) = \inf_{k \in \mathbb{N}} (\min_{y \in F_k} d(x, y)) = \inf_{k \in \mathbb{N}} 2^{-k} = 0$ .  $\square$

*Proof of Proposition 2.5.* Suppose first that, for some  $r > 0$ , there exists an infinite set  $E$  with the stated properties. If  $(X, d)$  is totally bounded then  $X$  is the union of a finite collection of balls each having radius  $r/2$ . At least one of these balls must contain infinitely many elements of  $E$ . But if two distinct elements  $x$  and  $y$  of  $E$  are in the same ball of radius  $r/2$  then they must satisfy  $d(x, y) \leq r$ . This is a contradiction which shows that  $(X, d)$  is not totally bounded.

For the converse implication, suppose that  $(X, d)$  is not totally bounded. Then there exists some  $r > 0$  such that  $X$  is not the union of any finite collection of balls of radius  $r$ . We will construct an infinite sequence  $\{x_n\}_{n \in \mathbb{N}}$  of points in  $X$  such that  $d(x_m, x_n) > r$  for all  $m, n \in \mathbb{N}$  with  $m \neq n$ . First choose  $x_1$  to be any point of  $X$ . If  $d(x_1, y) \leq r$  for every  $y \in X$  then  $X = B(x_1, r)$  which we know must be false. So there exists some point  $x_2 \in X$  such that  $d(x_1, x_2) > r$ . We will now use a recursive procedure to obtain the points  $x_n$  for all  $n > 2$ . Suppose we have already obtained  $k - 1$  points  $x_1, x_2, \dots, x_{k-1}$  in  $X$  such that

$$d(x_n, x_m) > r \text{ for all } m, n \in \{1, 2, \dots, k-1\} \text{ with } m \neq n. \quad (34)$$

If it were true that  $\min_{n \in \{1, 2, \dots, k-1\}} d(x_n, y) \leq r$  for each  $y \in X$  then this would imply that  $X = \bigcup_{n=1}^{k-1} B(x_n, r)$ , which we know to be false. Thus there exists at least one point  $y_* \in X$  which satisfies

$$\min_{n \in \{1, 2, \dots, k-1\}} d(x_n, y_*) > r. \quad (35)$$

If we choose  $x_k = y_*$  then it follows from (34) and (35) that we now have  $d(x_n, x_m) > r$  for all  $m, n \in \{1, 2, \dots, k\}$  with  $m \neq n$ .

Of course we now take  $E$  to be the set of all the points  $x_n$  and the proof is complete.  $\square$

*Proof of Proposition 2.6.* If  $(X, d)$  is not totally bounded then the infinite sequence  $\{x_n\}_{n \in \mathbb{N}}$  which can be constructed as in the second part of the proof of Proposition 2.5 clearly cannot have a Cauchy subsequence.

It remains to show the reverse implication. Suppose then that  $(X, d)$  is totally bounded and that  $\{x_n\}_{n \in \mathbb{N}}$  is an arbitrary sequence in  $X$ . Let  $E$  be the set which contains all elements of this sequence. If at least one element of  $E$  coincides with  $x_n$  for all  $n$  in some infinite subset  $W$  of  $\mathbb{N}$ , then obviously  $\{x_n\}_{n \in W}$  is a Cauchy subsequence of  $\{x_n\}_{n \in \mathbb{N}}$  and the proof is complete. Thus we may suppose that  $E$  contains infinitely many elements. We will now construct a sequence of balls  $\{B_k\}_{k \in \mathbb{N}}$  such that the following properties hold for each  $k \in \mathbb{N}$ :

- (i)  $B_k$  has radius  $2^{-k}$ , and
- (ii) the set  $\bigcap_{j=1}^k B_j$  contains infinitely many elements of the set  $E$ .

To begin this construction we simply use the fact that  $X$  is the union of a finite collection of balls of radius 1 and so at least one of these balls, which will be our  $B_1$ , must contain infinitely many elements of  $E$ . Now the construction of the  $B_k$ 's for  $k > 1$  can be done recursively. More specifically, suppose that we have constructed

$B_1, B_2, \dots, B_{m-1}$  with the stated properties (i) and (ii) for  $k = 1, 2, \dots, m-1$ . Again by (a) we have  $X = \bigcup_{p=1}^N V_p$  for some integer  $N$ , where each of the sets  $V_p$  is a ball of radius  $2^{-m}$ . Therefore  $\bigcap_{j=1}^{m-1} B_j = \bigcup_{p=1}^N \left( V_p \cap \bigcap_{j=1}^{m-1} B_j \right)$ . So, for at least one value of  $p$ , the set  $V_p \cap \bigcap_{j=1}^{m-1} B_j$  must contain infinitely many elements of  $E$ . We choose  $B_m$  to be  $V_p$  for that value of  $p$ , and thus we have completed the inductive step which is required to carry out the construction of  $B_k$  for all  $k \in \mathbb{N}$ .

Now we can very easily obtain, as required, a Cauchy subsequence  $\{x_{n_k}\}_{k \in \mathbb{N}}$  of the given sequence  $\{x_n\}_{n \in \mathbb{N}}$ . More explicitly, we are going to construct a strictly increasing sequence  $\{n_k\}_{k \in \mathbb{N}}$  of positive integers such that  $x_{n_k} \in \bigcap_{j=1}^k B_j$  for each  $k \in \mathbb{N}$ . First we choose  $n_1$  so that  $x_{n_1}$  is some element of  $E$  which is contained in  $B_1$ . Then we proceed inductively. Suppose that we have already obtained  $n_1, n_2, \dots, n_{k-1}$  such that  $x_{n_m} \in \bigcap_{j=1}^m B_j$  for  $m = 1, 2, \dots, k-1$  and  $n_1 < n_2 < \dots < n_{k-1}$ . Since  $\bigcap_{j=1}^k B_j$  contains infinitely many elements of  $E$ , there must exist some integer  $n_k$  which satisfies  $n_k > n_{k-1}$  and  $x_{n_k} \in \bigcap_{j=1}^k B_j$ . Finally it is easy to check that the elements  $x_{n_k}$  which we have constructed in this way for all  $k \in \mathbb{N}$  do indeed form a Cauchy sequence: Given any  $\epsilon > 0$ , choose  $N$  such that  $2^{-N+1} < \epsilon$ . Then if  $N \leq p < q$  we have that  $x_{n_p}$  and  $x_{n_q}$  are both contained in  $\bigcap_{j=1}^N B_j$  and therefore in  $B_N$ . Since  $B_N$  is a ball of radius  $2^{-N}$  it follows that  $d(x_{n_p}, x_{n_q}) < 2 \cdot 2^{-N} < \epsilon$ .  $\square$

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